

## THE ALGEBRA DETECTIVE: IF SNAPE IS A SNAKE, THEN P = K!

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Algebra is often introduced as arithmetic done with letters. In this article, we follow this analogy and see where it takes us. We ask questions, such as: If "SNAPE = SNAKE," can we conclude that "P = K"? Here is the spoiler: the path takes us into what mathematicians call group theory, a subfield of algebra that studies symmetry, and allows us to say interesting things about various languages like English and Swedish (and one very important fact about algebra itself).

### WHAT IS ALGEBRA REALLY?

Everybody is introduced to algebra at some point in school mathematics. Algebra comes to us gradually. First, we solve basic arithmetic problems. For example, we might be offered the expression:

$$3 + 5 = \square.$$

We think about our arithmetic facts and fill in the square with 8. Pretty soon, the square changes places and we see:

$$3 + \square = 8.$$

We know what is going on, and we say the square has to be 5 this time.

When teachers, textbook writers, and curriculum developers decide we are ready for it, the boxes disappear, getting replaced by letters. So, in a class called Algebra, a student might see:

$$3 + x = 8.$$

To solve this problem, one “isolates the variable,” applying the same mathematical operation on both sides of the equality sign (“subtract 3”). This might sound fancy, but the problem is actually the very same as the one before it. The box is now a letter; that is the only difference.

In some very basic sense then, school algebra is doing arithmetic with letters. This is actually an incomplete, and somewhat inaccurate, definition of algebra, but it will suffice for now.

### **IF ALGEBRA IS ABOUT LETTERS, WHAT CAN IT TELL US ABOUT ENGLISH?**

In algebra class, we are soon working with more and more complicated expressions involving variables and numbers together, such as:

$$xy + 10 = 16.$$

“Isolating the variables” gives us:

$$xy = 6.$$

If we are told that  $x = 3$ , we can plug in the value for  $x$  and see:

$$3y = 6.$$

Dividing both sides by 3 yields  $y = 2$ .

Eventually, we get really good at noticing common factors on both sides of the equality sign and canceling them out:  $4x = 16$  becomes  $x = 4$ ,  $11y = 33$  becomes  $y = 3$ , and so on.

Could we do all of algebra with only letters, the way we do arithmetic with only numbers? In that case, letters could even be actual letters,

not placeholders. What would it mean to say that two expressions are equal to one another in this scenario? For example, Professor Severus Snape of Hogwarts is a famous Slytherin, so it makes sense to say that "SNAPE = SNAKE." We could also say:

authors of this article = Per + Gizem,

or,

my favorite food = ice cream.

This last one is tricky, because "my" is always context dependent. For example, if Gizem is reading this sentence, "my favorite food" will mean "Gizem's favorite food" and the resulting sentence will be a true statement. However, if her sister were reading the same sentence, she would be asserting that Gizem's sister's favorite food is ice cream, which is definitely not a true statement (Yes, there are people who just do not like ice cream. Weird, we know!). One could even consider pronouns to be the ultimate variables. After all, each pronoun takes the place of a noun determined by the context of the earlier conversation.

Another thing we could do using only letters is to make two expressions equal if they sound the same. For example, in English, the letter collection "you" sounds the same as the single letter "u." So, we could set them equal to one another:

you = u.

Does your eager algebra eye want to do some cancellations? Do you think you can cancel the "u"s on both sides and get "yo = 1"? Why not? But then "dye = die" gives you "y = i" and so on. How far can you go? And just what does "y = i" mean?

## WHAT IS GOING ON? A GROUP THEORY PRIMER

In mathematics, group theory is the field of algebra that studies symmetry using certain algebraic structures called groups. A **group** is a bag of objects (see also [1]) together with a way of taking two objects from the bag to make another object that is also in the bag. The objects can be anything: numbers, words, or even ice cream cones. The "way" is called an **operation** and is often written " $\bullet$ ": if *object1* and *object2* are two objects in the bag,

$$\text{object1} \bullet \text{object2}$$

is the object in the bag we make from *object1* and *object2* with the operation " $\bullet$ ." We can form yet another object by first taking the object *object1*  $\bullet$  *object2* and then the object *object3* from the bag, to get the

### GROUP

A bag of objects together with an operation that takes two objects from the bag to make another object that is also in the bag. The operation needs to obey the associative law, and the special object *theone* must be in the bag. Last, all objects in the bag must have an inverse that is also in the bag.

### OPERATION

A way of taking two objects from a bag to form a new object that is also in the bag.

object

$$(\text{object1} \bullet \text{object2}) \bullet \text{object3}.$$

But what if we took instead *object1* first and then

$$\text{object2} \bullet \text{object3}?$$

In a group, the resulting object is the same! In other words, we require that what is called the **associative law** holds:

$$(\text{object1} \bullet \text{object2}) \bullet \text{object3} = \text{object1} \bullet (\text{object2} \bullet \text{object3}).$$

We also require that there is a special object in the bag, let us call it *theone*, since it is so special, such that for any object *object* in the bag,

$$\text{object} \bullet \text{theone} = \text{theone} \bullet \text{object} = \text{object}.$$

Our last requirement is that we should be able to cancel objects on both sides of “=” to get the resulting object *theone*. This can be expressed as follows: we want any object *object* in the bag to have a buddy in the bag called an **inverse**, say *inverseobject*, such that

$$\text{object} \bullet \text{inverseobject} = \text{inverseobject} \bullet \text{object} = \text{theone}.$$

That’s it! A bag of objects with an operation that satisfies these three requirements is called a group.

*Example 1:* We could take the bag of objects to be all the numbers (positive, negative, and zero) and “•” to be “+.” For example,  $3 + 4 = 7$  and  $7 + 2 = 9$ , so  $(3 + 4) + 2 = 7 + 2 = 9$ . We also have  $4 + 2 = 6$  and  $3 + 6 = 9$ , so  $3 + (4 + 2) = 3 + 6 = 9$ . In other words,  $(3 + 4) + 2 = 3 + (4 + 2)$ . This holds true for all numbers in our bag. Our special object *theone* turns out to be the number 0. Since  $7 + (-7) = 0$ , the inverse of 7 has to be  $-7$ . Can you figure out what the inverse of any number must be?

*Example 2:* Let us try to make a group out of the letter *x*. We start by putting *x* in our bag of objects. In order to make this a group, we also need to add the special object *theone* to our bag. Instead of writing *theone*, which is a bit cumbersome, we use the symbol 1. Last, we also need to add the inverse of *x*. We put a hat on *x* and simply say that this new letter,  $\hat{x}$ , is the inverse of *x*. This means that with *x*,  $\hat{x}$ , and 1 in our bag,  $x \bullet \hat{x} = 1$ . Is this a group? No, not yet. For example,  $x \bullet x$  also needs to be in our bag, so we simply add it and its inverse  $\hat{x} \bullet \hat{x}$  as well. Continuing like this and assuming that the associative law holds, we get the so-called group generated by *x*.

*Example 3:* Suppose that we take the group generated by *x* as described in Example 2 and add the rule  $x = 1$ . If we replace *x* by 1

### ASSOCIATIVE LAW

If “•” is an operation and *object1*, *object2*, and *object3* are objects in some bag, the associative law says that the object  $(\text{object1} \bullet \text{object2}) \bullet \text{object3}$  is the same object as  $\text{object1} \bullet (\text{object2} \bullet \text{object3})$ .

### INVERSE

If “•” is an operation and *object* an object in some bag, the inverse of *object*, let us call it *inverseobject*, is an object in the bag such that  $\text{object} \bullet \text{inverseobject} = \text{inverseobject} \bullet \text{object} = \text{theone}$ . Here, *theone* is the special object in the bag.

### TRIVIAL GROUP

The smallest group possible, the bag of objects containing only one object, *theone*.

in  $x \bullet \hat{x} = 1$ , we get  $1 \bullet \hat{x} = \hat{x} = 1$ . We also get  $x \bullet x = 1 \bullet 1 = 1$ . In fact, all objects in our bag must be equal to 1. In other words, our bag contains only one object, 1. From the definition, the bag in a group needs to have at least one object in it, *theone*. Therefore, this must be the smallest group possible, so it is called the **trivial group**.

**Example 4 | Let us explore a more visual example. Consider the Scrabble tile for the letter R:**

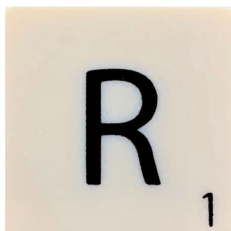


Photo: Per Bäck.  
This tile can be rotated  $90^\circ$  clockwise to give us

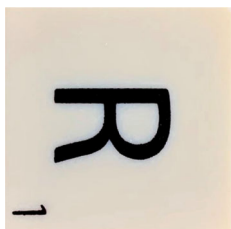


Photo: Per Bäck.  
If you rotate it  $90^\circ$  more, you get:

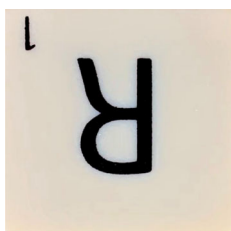


Photo: Per Bäck.  
Another  $90^\circ$  rotation gives you:



Photo: Per Bäck.  
and a final  $90^\circ$  rotation will get you back to the original tile. This is a nice example of a group. Each of these operations can be reconstructed from the single clockwise rotation by  $90^\circ$ . If we set  $x$  to be the "rotate clockwise by  $90^\circ$ " operation, then the group we get is the bag containing  $x$ ,  $x^2$ ,  $x^3$ , and  $x^4$ ; this last one is the same as *theone*, or in this context, the "do-nothing" operation. Do you see what the inverses are?

## HOMOPHONIC GROUPS

Can we make a group out of the whole alphabet? The English alphabet consists of 26 letters  $a, b, c, \dots, z$ . We invent the inverse English alphabet to be the letters of the English alphabet with hats on them,  $\hat{a}, \hat{b}, \hat{c}, \dots, \hat{z}$ . We form all possible words from the letters in the two alphabets and put them in a bag. By a word, we mean any combination of letters, so  $qw\hat{e}rt\hat{y}asdf$ ,  $brain$ ,  $d$ , and  $foo$  are all words in our bag. Whenever a letter lies immediately next to its inverse letter in a word, we cancel the two letters. This means that

$$fo\hat{o} = f$$

and

$$yo\hat{u}u = yo.$$

We also invent an operation  $\bullet$  on the bag of words to be that of gluing two words together to make a new word, so that

$$d \bullet foo = dfoo$$

and

$$foo \bullet d = food.$$

This also means that

$$a \bullet \hat{a} = a\hat{a} = ,$$

the word consisting of no letters at all—the **empty word**! Continuing, the operation  $\bullet$  obeys the associative law. For example,

$$f \bullet (oo \bullet d) = f \bullet ood = food$$

and

$$(f \bullet oo) \bullet d = foo \bullet d = food.$$

This almost looks like a group where the special object is the empty word! However, we need not only the letters to have inverses, but all the words as well. This works if we say that the inverse of a word is the word spelled backwards and with a hat on each letter. For example, the inverse of  $yo$  is  $\hat{o}\hat{y}$  since then

$$yo \bullet \hat{o}\hat{y} = yo\hat{o}\hat{y} = y\hat{y} = .$$

Just as in Example 2, we could use the symbol 1 for the special object (the empty word in this case), so that

$$y\hat{y} = 1.$$

### EMPTY WORD

The word with no letters in it.

## FREE GROUP

The group made from some alphabet by using the operation of gluing letters together to form words, and then gluing words together to form new words. In the bag of objects are all words possible, including the empty word, which is the special object *theone*.

## HOMOPHONIC GROUP

The free group on some alphabet where words that sound the same are equal.

The above group is called the **free group** on the English alphabet. Free, because it is the freest way one can make a group out of the English alphabet.

Now, suppose that we add the rule that two words are equal if they sound the same, as we suggested at the end of the second section. Then this means that

$$you = u.$$

We can cancel the “*u*”s by using the operation  $\bullet$  together with  $\hat{u}$  on each side. On the left-hand side, we get

$$you \bullet \hat{u} = you\hat{u} = yo,$$

and on the right-hand side

$$u \bullet \hat{u} = u\hat{u} = 1,$$

so  $yo = 1$ .

Words that sound the same are called homophones, and the free group on the English alphabet where homophones are equal is called the **homophonic group** of English. The word *you* and the letter *u* are therefore the same in the homophonic group of English, and by the above calculations, so are *yo* and the empty word (for which we used the symbol 1). Mathematicians have shown that the homophonic groups of English and French are the groups with just the empty word in them—the trivial group [2]. Other authors have shown that the homophonic group of German is the trivial group, while those of Korean and Turkish are not [3].

## SWEDISH, ANYONE?

Why not learn Swedish while doing group theory? In fact, we will see that it is trivial! The Swedish alphabet has 29 letters in it: the 26 letters that form the English alphabet together with the three funny-looking letters *å*, *ä*, and *ö*. By using a standard dictionary like [4], we will see that the homophonic group of the Swedish language is the trivial group. First, we use something that cannot be found in the Swedish Academy’s dictionary [4], but is well-known to all Swedish mathematicians, namely that *om* and *omm* are pronounced the same. In other words,

$$om = omm,$$

so by canceling *om* on both sides,  $m = 1$ . Continuing,

$$t = te, \quad jul = hjul, \quad v\ddot{a}rd = v\ddot{a}rld, \quad an = ann, \quad pol = pool,$$

so  $e = h = l = n = o = 1$ . We also have

$$\begin{aligned} \text{upplåst} &= \text{uppblåst}, & \text{hårdra} &= \text{hårdra}, & \text{bygd} &= \text{byggd}, \\ \text{saltare} &= \text{psaltare}, & \text{avisa} &= \text{avvisa}, \end{aligned}$$

which gives us  $b = d = g = p = v = 1$ . Since

$$\text{alf} = \text{alv}, \quad \text{jäst} = \text{gäst}, \quad \text{håv} = \text{hov}, \quad \text{bäck} = \text{beck},$$

we have  $f = v = 1, j = g = 1, å = o = 1, ä = e = 1$ . Now,

$$\text{board} = \text{bård}, \quad \text{sig} = \text{sej}, \quad \text{skälva} = \text{själva}, \quad r = \text{ärr},$$

and so  $a = 1a = oa = å = 1, i = i1 = ig = ej = 1, k = j = 1$ , and  $r = 1rr$  gives us  $r = 1$ . Moreover,

$$\text{chippa} = \text{kippa}, \quad \text{wienare} = \text{vinare}, \quad \text{gray} = \text{grej},$$

so we must have  $c = c1 = ch = k = 1, w = w1 = wie = vi = 1$ , and  $y = 1y = ay = ej = 1$ . Now,

$$\text{sensor} = \text{censor}, \quad \text{tjeck} = \text{check},$$

and so  $s = c = 1, t = t1 = tj = ch = 1$ . We have

$$\text{koxa} = \text{koksa}, \quad \text{zoo} = \text{so},$$

which gives us  $x = ks = 1, z = z1 = zo = s = 1$ . We also have

$$\text{sioux} = \text{so},$$

so we get  $u = 1u1 = sioux = so = 1$ . At last,

$$\text{toquen} = \text{token}, \quad \text{övre} = \text{oeuvre},$$

and therefore  $q = q1 = qu = ck = 1$  and  $ö = oeu = 1$ . We have shown that all the 29 letters must be equal to 1. Since the only word we can form from 1 is 1, we have shown that the homophonic group of Swedish is trivial!

## SO WHAT?

Why is all of this important? Here is an argument that might appeal to many mathy folks: Since any word in the trivial group is the empty word 1 and English is trivial, *algebra* = 1 and *awesome* = 1. Therefore, *algebra* = *awesome*. Now isn't that something?

On a more serious note, the homophonic group of a language tells us how well the alphabet represents what the language sounds like.



English, French, German, and Swedish, whose homophonic groups are all trivial, have many words where different letters are pronounced the same. Korean and Turkish, whose homophonic groups are non-trivial, have younger alphabets whose letters sound more similar to what they do in actual words. In other words, just playing around with letters in fun and seemingly meaningless ways can actually tell us something interesting about languages and their writing systems. Maybe the next time someone asks you what algebra is good for, you can tell them that it can help us understand something as human as language!

If you are excited by this article and want to learn more about groups, there are many books out there that might be good next steps. Alternatively, you might check out this interesting article on more examples of groups [5], or this neat online exploration of one of the most ambitious group theory projects of the twentieth century [6].

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## YOUNG REVIEWER



### GIOVANNI, AGE: 10

I was born in California and then lived in the Southwest for almost half my life, but now I am living in Virginia. I usually play videogames at home but I also do other things. I really like art and sketching. I won the state fair art contest. I want to be an architect or engineer when I grow up. I also do like writing stories, I am working on a story or book called "Within Our Walls."

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