



Perturbation Theory for Time-Dependent Quantum Systems Involving Complex Potentials

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We explore how to apply perturbation theory to complicated time-dependent Hamiltonian systems that involve complex potentials. To do this, we introduce a generalized time-dependent oscillator to which the complex potentials are connected through a weak coupling strength. We regard the complex potentials in the Hamiltonian as the perturbed terms. Quantum characteristics of the system, such as wave functions and expectation values of the Hamiltonian, are investigated on the basis of the perturbation theory. We apply our theory to particular systems with explicit choices of time-dependent parameters. Through such applications, the time behavior of the quantum wave packets and the spectrum of expectation values of the Hamiltonian are analyzed in detail. We confirm that the imaginary parts of expectation values of the Hamiltonian are not zero but very small, whereas the real parts deviate slightly from those of the unperturbed system.

Keywords: perturbation theory, time-dependent Hamiltonian system, complex potential, Schrödinger equation, wave function, expectation value

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1. INTRODUCTION

In the case that we are unable to derive exact quantum solutions for a perturbed system, perturbation theory is a useful tool for obtaining approximate quantum solutions. When we apply perturbation theory, it should be supposed that the scale of the perturbed terms in the potential is relatively small. The perturbation theory is valid only when the quantum solutions of the system in which the perturbation potentials have been removed are exactly known or derivable. The perturbation theory was originally developed for Hermitian systems in which the potential is real. Hence, in conventional quantum mechanics, the perturbation theory has, in large, been developed for the systems in which the potentials are real Hermitian that allows only the spectrum of real expectation values for quantum observables. Whereas the eigenvalue problem and the effects of perturbations on stationary Hamiltonian systems are well known in non-relativistic quantum physics, the perturbation techniques for the time-dependent Hamiltonian systems (TDHSs) with complex potentials have been investigated much less. This is partly due to the difficulty of mathematical procedures when we apply perturbation techniques in TDHSs.

Time-independent perturbation theory is a mathematical tool for treating quantum systems whose Hamiltonian involves small static perturbing terms which do not induce transitions to other quantum states. In a case that the Hamiltonian is a function of time, transitions between quantum states may take place. Then, we should consider time-dependent perturbation theory instead of time-independent one. Provided that the perturbing Hamiltonian is differentiable with respect to time in that case, the transition probability is determined from the time derivative of the perturbing Hamiltonian [1]. Hence, if the perturbing Hamiltonian is slowly varying, we can adopt adiabatic

theorem which assumes that the quantum system remains in its instantaneous eigenstate [2]. This allows the use of time-independent perturbation theory as an approximation in that situation. For more details of time-dependent perturbation theory and its relevance to time-independent perturbation theory, refer to Mandal and Hunt [1, 3] and Langhoff et al. [4].

On account of recent attention to the quantum problem of physical systems characterized by complex potentials, the necessity for the extension of perturbation theory to complex potential systems has gradually emerged. The data in some of the processes of elastic scattering such as nucleus-nucleus scattering [5] and electron scattering from solid crystals [6] fit better if one adopts complex potentials involving a non-positive imaginary part. Moreover, according to the advancement of techniques for unfolding quantum theory, the necessity of quantum manipulation with complex potentials takes place in many different areas of physics and related topics such as optics [7–11], statistical physics [12, 13], relativistic quantum field theory [14–16], nuclear physics [17], condensed matter physics [18–20], and biology [21]. To analyze the stability and nonlinear modes of optical beams propagating along an asymmetric waveguide, an imaginary potential has been adopted [11]. The structures of the energy band in elastic scattering processes governed by a periodic complex potential have also been studied [22].

According to such a research trend, we will investigate, in this work, how to apply perturbation theory to complex potential systems. We will adopt a general Hamiltonian that corresponds to a time-dependent harmonic oscillator to which complex potentials are coupled as perturbation terms. We will consider adiabatic evolution [23–25] of the states under slow variations of parameters while the magnitudes of perturbation terms are quite small compared to the unperturbed term at all times. Then, there will be no transition between eigenstates and, as a consequence, we can use usual time-independent perturbation theory. The wave functions corrected by perturbed potentials will be analytically evaluated. Using the wave functions involving correction terms, quantum characteristics of the system will be investigated. Furthermore, the spectrum of the expectation values of the Hamiltonian (EVH) will be rigorously evaluated. We will apply our development to some particular cases and analyze the time evolution of the corresponding wave functions and the EVH.

2. THEORETICAL FRAMEWORK FOR A COMPLEX POTENTIAL SYSTEM

In this section, we outline the quantum structure for a TDHS which involves complex potentials as perturbations to the system. The equation of motion for lots of actual physical models in quantum mechanics are described not only by linear terms which allow us to have exact mathematical solutions, but also by nonlinear terms associated with interactions of the system with various subsystems. If the coupling parameters of the nonlinear parts are sufficiently small compared to others, we can treat them as perturbation terms. Regarding this, we consider the system of

which Hamiltonian is represented as

$$\hat{\mathcal{H}} = \hat{H} + \epsilon \hat{H}_p, \quad (1)$$

where \hat{H} is a time-dependent quadratic Hamiltonian and \hat{H}_p is a perturbing Hamiltonian, while ϵ is a small coupling parameter. In this case, the perturbation theory is available only when the second term on the right hand side of Equation (1) is very small relative to the first term. We assume that \hat{H} and \hat{H}_p are given by

$$\hat{H} = f(t) \frac{\hat{p}^2}{2m_0} + g(t)(\hat{q}\hat{p} + \hat{p}\hat{q}) + \frac{1}{2}h(t)m_0\omega^2(t)\hat{q}^2, \quad (2)$$

$$\hat{H}_p = \sum_j k_{R,j}(t)\hat{q}^j + i \sum_l k_{I,l}(t)\hat{q}^l, \quad (3)$$

where $f(t)$, $g(t)$, $h(t)$, $k_{R,j}(t)$, and $k_{I,l}(t)$ are time functions that are differentiable with respect to time, and $f(t) \neq 0$. m_0 is the mass and $\omega(t)$ is a time-dependent angular frequency. Although this is a one-dimensional Hamiltonian, we can apply it to various different kinds of oscillators depending on the choice of the explicit formulae of time functions. As examples for such, we will apply it to two special cases at later by choosing the time functions being particular forms. In the case where the mass of the system varies over time, we can regard m_0 as the initial mass. Notice that the additional term \hat{H}_p in the Hamiltonian makes the system be an anharmonic oscillator.

The Schrödinger solutions of the system are different depending on the choice of the time functions in the Hamiltonian. If we consider a discrete spectrum of solutions, the Schrödinger equation for the overall Hamiltonian $\hat{\mathcal{H}}$ can be written as

$$i\hbar\partial\Psi_n(q,t)/\partial t = \hat{\mathcal{H}}\Psi_n(q,t). \quad (4)$$

To apply the perturbation theory at this stage, it is necessary to know complete quantum solutions associated with the unperturbed Hamiltonian \hat{H} as mentioned in the introductory part. For this reason, we first derive exact solutions of the Schrödinger equation for \hat{H} from

$$i\hbar\partial\psi_n(q,t)/\partial t = \hat{H}\psi_n(q,t). \quad (5)$$

Let us take a moment to review the theory for solving this equation exactly. Because \hat{H} involves time functions, it is very difficult to solve this equation relying on the conventional method. For such TDHSs, it is known based on the invariant operator theory that the Schrödinger solutions are expressed in terms of the solutions of a classical equation associated with the system [26, 27]. Hence, let us consider a time function $\rho(t)$ which is the solution of the following nonlinear equation

$$\ddot{\rho}(t) + \omega_\rho^2(t)\rho(t) - \frac{\Omega^2}{\rho^3(t)} = 0, \quad (6)$$

where

$$\omega_\rho^2(t) = f(t)h(t)\omega^2(t) + \frac{2\dot{f}(t)g(t)}{f(t)} + \frac{\ddot{f}(t)}{2f(t)} - \frac{3\dot{f}^2(t)}{4f^2(t)} - 4g^2(t) - 2\dot{g}(t). \quad (7)$$

Then, we can define the annihilation operator of the system in terms of $\rho(t)$ as

$$\hat{a} = \sqrt{\frac{1}{2\hbar\Omega}} \left[\sqrt{\frac{m_0}{f(t)}} \left(\frac{\Omega}{\rho(t)} + iG(t) \right) \hat{q} + i\sqrt{\frac{f(t)}{m_0}} \rho(t) \hat{p} \right], \quad (8)$$

where Ω is an arbitrary real positive constant, and

$$G(t) = \left(2g(t) - \frac{\dot{f}(t)}{2f(t)} \right) \rho(t) - \dot{\rho}(t). \quad (9)$$

Of course, the Hermitian adjoint of Equation (8), \hat{a}^\dagger , is the creation operator. These ladder operators obey the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. The time evolution of $\hat{a}(t)$ can be simply represented as $\hat{a}(t) = \hat{a}(0)e^{-i[\eta(t) - \eta(0)]}$, where

$$\eta(t) = \Omega \int_0^t \frac{d\tau}{\rho^2(\tau)} + \eta(0). \quad (10)$$

It is also possible to define an invariant operator using the ladder operators, such that [27, 28]

$$\hat{I} = \hbar\Omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \quad (11)$$

We can check that the direct differentiation of \hat{I} with respect to time results in zero, which means that $\hat{I}(t)$ is a time-invariant. If we write the eigenfunctions of \hat{I} as $\phi_n(q, t)$, they can be derived from the eigenvalue equation:

$$\hat{I}\phi_n(q, t) = \lambda_n \phi_n(q, t). \quad (12)$$

This is analogous to the time-independent Schrödinger equation in a stationary system. We can easily solve this equation from a minor evaluation; the exact results are given in **Appendix A**. According to the invariant operator theory, the wave functions of the unperturbed system are expressed in terms of $\phi_n(q, t)$ as [27, 28]

$$\psi_n(q, t) = \phi_n(q, t)e^{i\theta_n(t)}, \quad (13)$$

where

$$\theta_n(t) = -(n + 1/2)\eta(t) + \tilde{\theta}_n, \quad (14)$$

while $\tilde{\theta}_n$ is an arbitrary constant phase. Because $\phi_n(q, t)$ given in **Appendix A** are described by $\rho(t)$, the wave functions are also represented in terms of it. In fact, the reason why $\rho(t)$ has been introduced is that it is necessary in the expression of the quantum solutions of the unperturbed system of which Hamiltonian \hat{H} is dependent on time.

We can easily check that the ladder operators yield

$$\hat{a}|\phi_n(t)\rangle = \sqrt{n}|\phi_{n-1}(t)\rangle, \quad (15)$$

$$\hat{a}^\dagger|\phi_n(t)\rangle = \sqrt{n+1}|\phi_{n+1}(t)\rangle. \quad (16)$$

The quantum theory for a TDHS represented above will be used in the subsequent section in order to develop the perturbation theory of the system associated with the complex potentials.

3. PERTURBATION THEORY WITH THE COMPLEX POTENTIALS

Although perturbation theory does not give the spectrum of exact analytical quantum solutions for a dynamical system, it enables us to solve difficult problems in quantum mechanics from a series of routine calculations. If we regard that we cannot derive exact Schrödinger solutions in many cases of quantum mechanical problems, perturbation theory is quite useful in quantum mechanics. We will show how to manage perturbation theory for the case where the Hamiltonian involves time-dependent complex potentials on the basis of the associated theory represented in the previous section.

By expressing Equation (8) and its Hermitian adjoint \hat{a}^\dagger inversely, we obtain the representation of canonical variables \hat{q} and \hat{p} in terms of \hat{a} and \hat{a}^\dagger . Then we can easily derive the expectation values of \hat{H} using Equations (13), (15), and (16), leading to

$$\langle \psi_n | \hat{H} | \psi_n \rangle = \hbar W(t) \left(n + \frac{1}{2} \right), \quad (17)$$

where

$$W(t) = \frac{1}{2\Omega} \left[G^2(t) + \frac{\Omega^2}{\rho^2(t)} + f(t)h(t)\omega^2(t)\rho^2(t) - 4g(t)\rho(t)G(t) \right]. \quad (18)$$

The purpose of using perturbation techniques is to find the effects of small perturbed potentials on the whole system. In general, perturbation expansion gives a power series representation of a resultant quantity with respect to the perturbing parameter ϵ . The results of a calculation considering perturbations agree quite well with experimental data, but entails an infinite number of minor terms. However, the higher orders of such minor terms can, in general, be negligible from the perturbation corrections because their numeric scales are small enough.

Our theory is based on time-independent perturbation theory that is usually managed in terms of the eigenstates where the phases are not considered [29], as mentioned earlier. Hence, if we do not care about the minor phases of the wave functions contributed from the perturbing Hamiltonian, we can represent the overall wave functions of the system as

$$\langle q | \Psi_n \rangle = \langle q | \Phi_n \rangle e^{i\theta_n(t)}, \quad (19)$$

where $\langle q | \Phi_n \rangle$ are eigenstates and $\theta_n(t)$ are the phases. According to the perturbation theory, the eigenstates take the form

$$\begin{aligned} \langle q | \Phi_n \rangle &= \langle q | \phi_n \rangle \\ &+ \epsilon \sum_{n', (n' \neq n)} \Pi_1 \langle q | \phi_{n'} \rangle + \epsilon^2 \sum_{n', (n' \neq n)} \Pi_2 \langle q | \phi_{n'} \rangle + \dots, \end{aligned} \quad (20)$$

where $\langle q | \phi_n \rangle = \phi_n(q, t)$, which are given in **Appendix A**, and

$$\Pi_1 = \frac{\langle \phi_{n'} | \hat{H}_p | \phi_n \rangle}{\langle \phi_n | \hat{H} | \phi_n \rangle - \langle \phi_{n'} | \hat{H} | \phi_{n'} \rangle}, \quad (21)$$

$$\Pi_2 = \Pi_2^{(1)} + \Pi_2^{(2)}, \quad (22)$$

with

$$\Pi_2^{(1)} = \sum_{n'', (n'' \neq n)} \frac{\langle \phi_{n'} | \hat{H}_p | \phi_{n''} \rangle \langle \phi_{n''} | \hat{H}_p | \phi_n \rangle}{(\langle \phi_n | \hat{H} | \phi_n \rangle - \langle \phi_{n'} | \hat{H} | \phi_{n'} \rangle)(\langle \phi_n | \hat{H} | \phi_n \rangle - \langle \phi_{n''} | \hat{H} | \phi_{n''} \rangle)}, \quad (23)$$

$$\Pi_2^{(2)} = -\frac{\langle \phi_{n'} | \hat{H}_p | \phi_n \rangle \langle \phi_n | \hat{H}_p | \phi_n \rangle}{(\langle \phi_n | \hat{H} | \phi_n \rangle - \langle \phi_{n'} | \hat{H} | \phi_{n'} \rangle)^2}. \quad (24)$$

By perturbation theory, the EVH are represented as

$$\langle \Phi_n | \hat{\mathcal{H}} | \Phi_n \rangle = \epsilon^0 \Lambda_0 + \epsilon^1 \Lambda_1 + \epsilon^2 \Lambda_2 + \dots, \quad (25)$$

where

$$\Lambda_0 = \langle \phi_n | \hat{H} | \phi_n \rangle, \quad (26)$$

$$\Lambda_1 = \langle \phi_n | \hat{H}_p | \phi_n \rangle, \quad (27)$$

$$\Lambda_2 = \sum_{n', (n' \neq n)} \frac{\langle \phi_n | \hat{H}_p | \phi_{n'} \rangle \langle \phi_{n'} | \hat{H}_p | \phi_n \rangle}{\langle \phi_n | \hat{H} | \phi_n \rangle - \langle \phi_{n'} | \hat{H} | \phi_{n'} \rangle}. \quad (28)$$

Up until now, we have developed perturbation theory for the general time-dependent Hamiltonian system which involves complex potentials. In order to see the applicability of our theory to particular systems, let us consider the case where $k_{R,1}(t) \neq 0$ and $k_{I,3}(t) \neq 0$, while all other $k_{R,j}(t)$ and $k_{I,l}(t)$ are zero. In this case, Equation (3) can be written in the form

$$\hat{H}_p = k_{R,1}(t)\hat{q} + ik_{I,3}(t)\hat{q}^3. \quad (29)$$

Now, from Equations (20)–(22), the corresponding wave functions are derived to be

$$\langle q | \Phi_n \rangle = \langle q | \phi_n \rangle + \epsilon \tilde{\Pi}_1 + \epsilon^2 \tilde{\Pi}_2 + \dots, \quad (30)$$

where, $\tilde{\Pi}_1$ is given by

$$\tilde{\Pi}_1 = \frac{1}{\hbar W(t)} [A(t)k_{R,1}(t)\varphi_{R,1} + iA^3(t)k_{I,3}(t)\varphi_{I,3}], \quad (31)$$

with

$$A(t) = \sqrt{\frac{\hbar f(t)}{2\Omega m_0}} \rho(t), \quad (32)$$

$$\varphi_{R,1} = \sqrt{n} \langle q | \phi_{n-1} \rangle - \sqrt{n+1} \langle q | \phi_{n+1} \rangle, \quad (33)$$

$$\begin{aligned} \varphi_{I,3} = & \frac{\sqrt{n(n-1)(n-2)}}{3} \langle q | \phi_{n-3} \rangle \\ & - \frac{\sqrt{(n+1)(n+2)(n+3)}}{3} \langle q | \phi_{n+3} \rangle \\ & + 3n^{3/2} \langle q | \phi_{n-1} \rangle - 3(n+1)^{3/2} \langle q | \phi_{n+1} \rangle. \end{aligned} \quad (34)$$

Because $\tilde{\Pi}_2$ in Equation (30) is a somewhat complicated form, we have represented it in **Appendix B**.

On the other hand, the EVH become

$$\begin{aligned} \langle \Phi_n | \hat{\mathcal{H}} | \Phi_n \rangle = & \hbar W(t) \left(n + \frac{1}{2} \right) \\ & + \frac{\epsilon^2}{\hbar W(t)} [A^6(t)k_{I,3}^2(t)(30n^2 + 30n + 11) \\ & - A^2(t)k_{R,1}^2(t) \\ & - 6iA^4(t)k_{R,1}(t)k_{I,3}(t)(2n + 1)] + \dots \end{aligned} \quad (35)$$

If large-scale perturbation expansions regarding higher order terms are required in the solutions, it is necessary to take the aid of computer algebra [30]. The last term in Equation (35) is an imaginary part of the energy. We can elucidate the time-dependent evolution of the probability density in a quantum state based on non-Hermitian quantum theory, where imaginary terms of the energy levels take place [10, 31–34] as well as the real ones. The novel outcome of non-Hermitian analyses is excluded in the problem of conventional quantization methods relevant to Hermitian Hamiltonians. The eigenvalues of Hermitian Hamiltonians and/or Pseudo Hermitian Hamiltonians are always real from a mathematical point of view. However, some systems/subsystems can violate the requirement of Hermiticity of Hamiltonians [35]. As can be seen from Equation (35), the expectation values of the Hamiltonian are complex. The possible existence of imaginary energy has a considerable interest from several decades ago [33]. Zhu and Cukier [34] adopted an imaginary energy method in order to formulate quantum rate theories. They have obtained a rate constant that can be used to explain the WKB quantum tunneling rate at low temperature and the quantum state transition rate at high temperature. Further, they have shown that their quantum rate constant has the same structure as the Miller's theory [36, 37] of quantum transition state at high temperature limit. Meanwhile, imaginary energy gaps between the complex eigenvalues of a system have been used to analyze cooperative effects such as superconductivity and superradiance [32]

Now let's turn to a special case where the unperturbed Hamiltonian corresponds to the Caldirola-Kanai oscillator [38, 39]:

$$\hat{H} = e^{\beta t} \frac{\hat{P}^2}{2m_0} + \frac{1}{2} e^{-\beta t} m_0 \omega_0^2 \hat{q}^2, \quad (36)$$

which can be obtained from the choice of time functions as $f(t) = h^{-1}(t) = e^{\beta t}$, $g(t) = 0$, and $\omega(t) = \omega_0$, where β and ω_0 are real positive constants. In addition, we choose

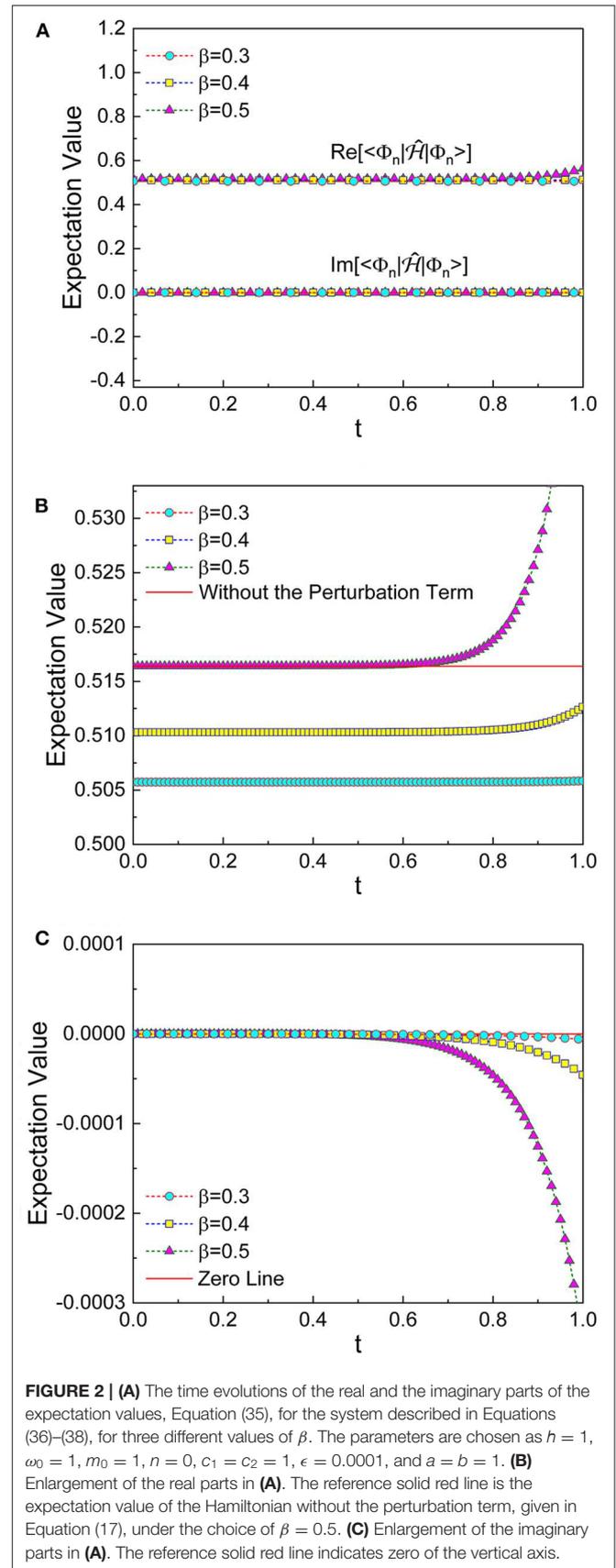
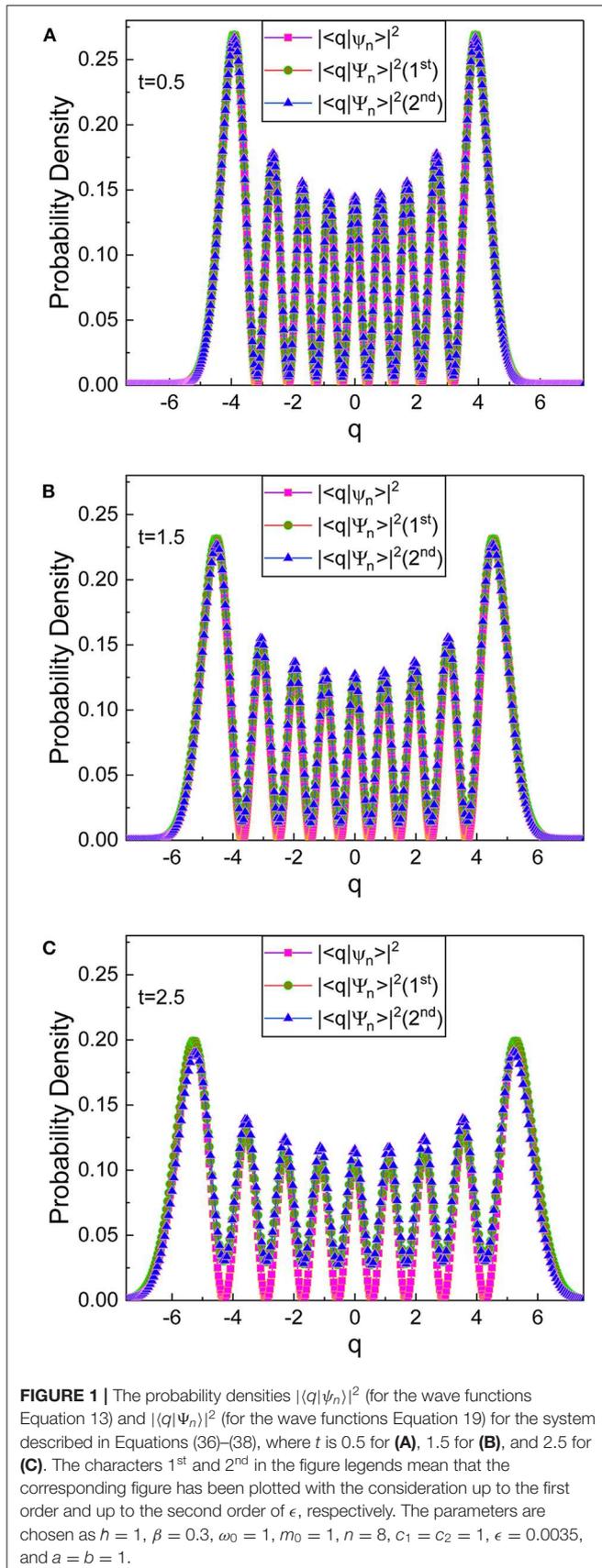
$$k_{R,1}(t) = a \quad k_{I,3}(t) = b, \quad (37)$$

where a and b are real constants. Then, the solution of the nonlinear equation, Equation (6), is given by

$$\rho(t) = [c_1 \cos^2(\bar{\Omega}t) + c_2 \sin^2(\bar{\Omega}t)]^{1/2}, \quad (38)$$

provided that $\Omega = (c_1 c_2)^{1/2} \bar{\Omega}$, where $\bar{\Omega} = (\omega_0^2 - \beta^2/4)^{1/2}$, and c_1 and c_2 are arbitrary real positive constants.

We have depicted the probability density $|\langle q | \Psi_n \rangle|^2$ for the system described above in **Figure 1**. Due to the choice of



$c_1 = c_2 = 1$ in this figure, $\rho(t)$ is a uniform function having the value 1. This figure shows that the deviation of the probability densities for the perturbed wave functions, Equation (19), at an initial time can be negligible and, as a consequence, they are almost the same as $|\langle q|\psi_n\rangle|^2$ which are probabilities associated to the unperturbed wave functions. However, such deviation increases gradually as time goes by. We can also confirm from this figure that $|\langle q|\Psi_n\rangle|^2$ considered ϵ up to the second order are nearly the same as those considered ϵ up to the first order. This means that we can neglect the terms which are higher order in ϵ in the wave functions, as expected.

Figure 2 shows the time evolution of the real and the imaginary parts of the expectation values of the Hamiltonian $\hat{\mathcal{H}}$ considering up to the second order of ϵ for the system described by Equations (36)–(38). From **Figure 2A**, we can confirm that the deviation of both the real and the imaginary parts of the EVH from those of the unperturbed Hamiltonian is, largely, not so significant, but is not zero. **Figures 2B,C** are enlarged graphs of the real and the imaginary parts of $|\langle \Phi_n|\hat{\mathcal{H}}|\Phi_n\rangle|^2$, respectively. If β is large, the expectation values of the real part of the Hamiltonian are also large. Real parts deviate toward the positive direction whereas imaginary parts toward the negative one and such deviations increase over time.

Let us show that the results, Equations (30)–(35), become well-known ones for a particular case. For this purpose, we put $\beta = 0$ from Equation (36) and $b = 0$ from Equation (37) while $a \neq 0$. Then, the system becomes a simple one that is described by the Hamiltonian

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m_0} + \frac{1}{2}m_0\omega_0^2\hat{q}^2 + \delta\hat{q}, \quad (39)$$

where $\delta = \epsilon a$. Then, under the choice $c_1 = c_2 = 1$, we have $\rho = 1$, $W = \Omega = \omega_0$, $G = 0$, and $A = [\hbar/(2m_0\omega_0)]^{1/2}$. From these, we easily confirm that Equations (30) and (35) reduce in this situation to

$$\begin{aligned} \langle q|\Phi_n\rangle &= \langle q|\phi_n\rangle + \frac{\delta}{\omega_0} \sqrt{\frac{1}{2\omega_0 m_0 \hbar}} (\sqrt{n}\langle q|\phi_{n-1}\rangle \\ &\quad - \sqrt{n+1}\langle q|\phi_{n+1}\rangle) \\ &\quad + \frac{\delta^2}{4\hbar m_0 \omega_0^3} (\sqrt{n+1}\sqrt{n+2}\langle q|\phi_{n+2}\rangle \\ &\quad + \sqrt{n}\sqrt{n-1}\langle q|\phi_{n-2}\rangle) + \dots, \end{aligned} \quad (40)$$

$$\langle \Phi_n|\hat{\mathcal{H}}|\Phi_n\rangle = \hbar\omega_0 \left(n + \frac{1}{2} \right) - \frac{\delta^2}{2m_0\omega_0^2} + \dots, \quad (41)$$

where

$$\langle q|\phi_n\rangle = \sqrt{\frac{m_0\omega_0}{\hbar\pi}} \frac{1}{\sqrt{2^n n!}} H_n(Q) e^{-Q^2/2}, \quad (42)$$

with $Q = \sqrt{m_0\omega_0/\hbar}q$. The results Equations (40) and (41) are well-known in the literature [see, for example [40]]. If we

choose $c_1 \neq 1$ and/or $c_2 \neq 1$, the quantum solutions for the unperturbed Hamiltonian \hat{H} are somewhat different from the standard ones and the result given in Equation (41) becomes different; however, the quantum solutions for \hat{H} still satisfy the Schrödinger equation in that case and the corresponding solutions approximated by the perturbation theory for the entire Hamiltonian are also allowed. Although we have chosen a simple reduced Hamiltonian, Equation (39), in order to show that our results become well-known ones for a particular case, the quantum solutions of the system described by this Hamiltonian can also be derived completely without using perturbation theory.

Now let us see another special case where the unperturbed Hamiltonian is given by

$$\hat{H} = \frac{\hat{p}^2}{2m_0} + \frac{1}{2}m_0\omega_0^2 t^4 \hat{q}^2. \quad (43)$$

This corresponds to the case where the time functions are given by $f(t) = h(t) = 1$, $g(t) = 0$, and $\omega^2(t) = \omega_0^2 t^4$. Meanwhile, we put the time functions in Equation (29) as

$$k_{R,1}(t) = e^{-\alpha t} \quad k_{I,3}(t) = c + d \cos(\gamma t), \quad (44)$$

where $c \geq d$. Then the solution of the nonlinear equation given in Equation (6) can be written in the form

$$\rho(t) = [\rho_1^2(t) + \rho_2^2(t)]^{1/2}, \quad (45)$$

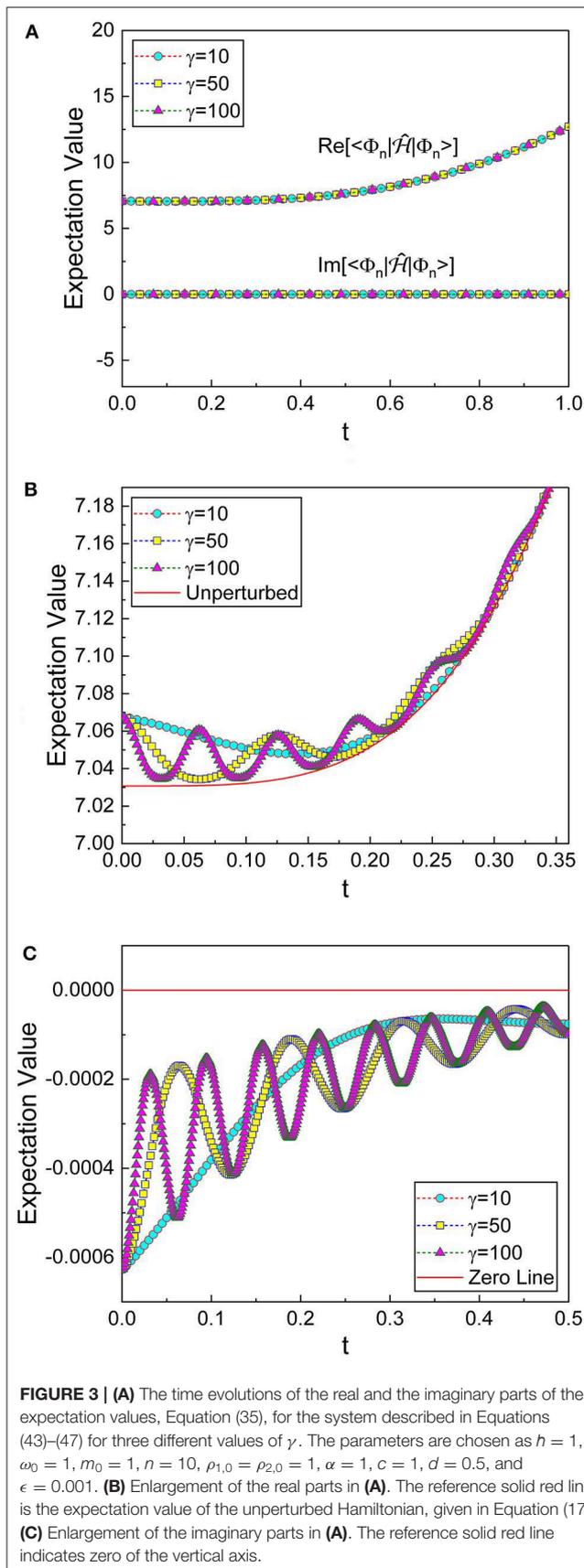
with

$$\rho_1(t) = \rho_{1,0} t^{1/2} J_\nu \left(\frac{1}{3} \omega_0 t^3 \right), \quad (46)$$

$$\rho_2(t) = \rho_{2,0} t^{1/2} N_\nu \left(\frac{1}{3} \omega_0 t^3 \right), \quad (47)$$

provided that $\Omega = \rho_1(t)\dot{\rho}_2(t) - \dot{\rho}_1(t)\rho_2(t)$, where $\rho_{1,0}$ and $\rho_{2,0}$ are real positive constants, J_ν and N_ν are Bessel functions of the first and second kind, respectively, while $\nu = 1/6$.

In **Figure 3**, we have plotted the EVH for the system described in Equations (43)–(47) considering up to the second order of ϵ . The imaginary part of the EVH does not vary much over time, whereas the real part increases as time goes by. This shows that the imaginary part is small but not zero, while the real part does not significantly deviate from the unperturbed one. We have enlarged the real and the imaginary parts of the expectation value in **Figures 3B,C**, respectively. It is interesting that both the real and the imaginary parts of the EVH oscillate over time, provided that γ is sufficiently large. The period of such oscillation becomes short as the angular frequency γ increases. In the case of $\alpha \rightarrow \infty$, we have $k_{R,1}(t) \rightarrow 0$, while the spectrum of the EVH is real and positive as conjectured by Bessis from numerical studies [see [41] and references therein].



4. CONCLUSION

The properties of the quantum states for perturbed Hamiltonian systems involving complex potentials were investigated. We have considered time-dependence of the imaginary part of the perturbing potentials as well as that of the real part, which are both coupled to a generalized harmonic oscillator through a weak coupling constant. The solutions of the Schrödinger equation of the system in the Fock state have been obtained using the perturbation theory. The effects of the perturbation on time behavior of the system have been analyzed. The perturbation corrections on the wave functions and on the EVH have been analytically investigated. Because we have considered complicated time-dependent Hamiltonian systems as a generalization, the EVH for each particular case is nonconservative. However, when we remove the time dependence of the Hamiltonian, our quantum solutions reduce to stationary states where the corresponding energy spectrum is conservative.

We have applied our theory to particular cases, such as the perturbed systems of which the unperturbed part is described by the Caldirola-Kanai Hamiltonian, and by a potential in which the angular frequency increases in proportion to the square of time. We see that the deviation of the probability densities and the EVH from those of the unperturbed system is, in large, not significant due to the weakness of the coupling. However, the deviation of the probability densities from those of the unperturbed system becomes large with the lapse of time. The imaginary part of the EVH is very small, but not zero, whereas the real part deviates slightly from that of the unperturbed system. In the case where the unperturbed part of the Hamiltonian corresponds to the Caldirola-Kanai oscillator, such deviation increases as the damping factor β grows. We have confirmed that the EVH for the system whose angular frequency is proportional to the square of t oscillate over time due to the time-variation of the chosen perturbing term in the Hamiltonian. The period of such oscillation becomes short as the angular frequency γ that appears in the perturbing term increases.

The development of the perturbation theory represented here may also be possibly applied to diverse different quantum systems which contain time-dependent complex perturbation potentials beyond those studied in the text. Some insights for characterizing complex potential systems are necessary in the light of the fact that a complex potential is receiving due attention in analyzing actual physical states, such as elastic scattering processes [5, 6], gain and loss in waveguides [11], band structures [22], and tunneling in absorptive media [42]. In a unified description of resonance and decay phenomena in open quantum systems, imaginary energy implies decay widths of resonance states, while decay widths are inversely proportional to resonance lifetimes [31, 43]. On the other hand, in a description of quantum and/or classical waves propagating across a complex potential, the imaginary energy plays the role of gain which is the degree of amplification of the transmitted waves [10]. Model-independent investigations of usual and

unusual quantum features of complex potential systems may be a very interesting topic for subsequent research as a further generalization of conventional quantum mechanics. In addition, PT based study for a perturbation theory of time-dependent Hamiltonian systems may also be a good research topic as a next task.

DATA AVAILABILITY STATEMENT

The datasets generated for this study are available on request to the corresponding author.

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The author confirms being the sole contributor of this work and has approved it for publication.

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Conflict of Interest: The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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APPENDIX A: SOLUTIONS OF EQUATION (12)

The eigenstates of the invariant operator for the system are obtained by directly solving Equation (12) or by using the properties of Equations (15) and (16) [28, 44, 45]. In our case, the results are given by

$$\phi_n(q, t) = \left(\frac{m_0 \Omega}{f(t) \rho^2(t) \hbar \pi} \right)^{1/4} (2^n n!)^{-1/2} H_n \left[\left(\frac{m_0 \Omega}{f(t) \rho^2(t) \hbar} \right)^{1/2} q \right] \times \exp \left[-\frac{m_0}{2f(t) \rho(t) \hbar} \left(\frac{\Omega}{\rho(t)} + iG(t) \right) q^2 \right], \quad (\text{A1})$$

$$\lambda_n = \hbar \Omega \left(n + \frac{1}{2} \right). \quad (\text{A2})$$

APPENDIX B: THE FORMULA OF $\tilde{\Pi}_2$ IN EQUATION (30)

$\tilde{\Pi}_2$ is obtained by evaluating the summation $\tilde{\Pi}_2 = \sum_{n', (n' \neq n)} \Pi_2 \langle q | \phi_{n'} \rangle$. Since Π_2 given in Equation (22) is composed of two terms, let us express $\tilde{\Pi}_2$ in the form

$$\tilde{\Pi}_2 = \tilde{\Pi}_2^{(1)} + \tilde{\Pi}_2^{(2)}, \quad (\text{B1})$$

where

$$\tilde{\Pi}_2^{(1)} = \sum_{n', (n' \neq n)} \Pi_2^{(1)} \langle q | \phi_{n'} \rangle, \quad (\text{B2})$$

$$\tilde{\Pi}_2^{(2)} = \sum_{n', (n' \neq n)} \Pi_2^{(2)} \langle q | \phi_{n'} \rangle. \quad (\text{B3})$$

The second term $\tilde{\Pi}_2^{(2)}$ is zero because the expectation values $\langle \phi_n | \hat{H}_p | \phi_n \rangle$ which are involved in it are zero. On the other hand, a rigorous evaluation of the first term using Equation (23) gives

$$\begin{aligned} \tilde{\Pi}_2^{(1)} = & \frac{A^2(t)}{\hbar^2 W^2(t)} \left\{ \frac{i}{3} A^2(t) k_{I,3} \sqrt{(n+1)(n+2)(n+3)} \right. \\ & \times \left(\frac{i}{6} A^2(t) k_{I,3} \sqrt{(n+4)(n+5)(n+6)} \langle q | \phi_{n+6} \rangle \right. \\ & + \frac{1}{4} [k_{R,1} \sqrt{n+4} + 3iA^2(t) k_{I,3} (n+4)^{3/2}] \langle q | \phi_{n+4} \rangle \\ & + \left. \frac{1}{2} [k_{R,1} \sqrt{n+3} + 3iA^2(t) k_{I,3} (n+3)^{3/2}] \langle q | \phi_{n+2} \rangle \right) \\ & + \frac{i}{3} A^2(t) k_{I,3} \sqrt{n(n-1)(n-2)} \\ & \times \left(\frac{i}{6} A^2(t) k_{I,3} \sqrt{(n-3)(n-4)(n-5)} \langle q | \phi_{n-6} \rangle \right. \\ & + \frac{1}{2} [k_{R,1} \sqrt{n-2} + 3iA^2(t) k_{I,3} (n-2)^{3/2}] \langle q | \phi_{n-2} \rangle \\ & + \left. \frac{1}{4} [k_{R,1} \sqrt{n-3} + 3iA^2(t) k_{I,3} (n-3)^{3/2}] \langle q | \phi_{n-4} \rangle \right) \\ & + [k_{R,1} \sqrt{n+1} + 3iA^2(t) k_{I,3} (n+1)^{3/2}] \\ & \times \left(\frac{i}{4} A^2(t) k_{I,3} \sqrt{(n+2)(n+3)(n+4)} \langle q | \phi_{n+4} \rangle \right. \\ & - \left. \frac{i}{2} A^2(t) k_{I,3} \sqrt{(n+1)n(n-1)} \langle q | \phi_{n-2} \rangle \right. \\ & + \left. \frac{1}{2} [k_{R,1} \sqrt{n+2} + 3iA^2(t) k_{I,3} (n+2)^{3/2}] \langle q | \phi_{n+2} \rangle \right) \\ & + [k_{R,1} \sqrt{n} + 3iA^2(t) k_{I,3} n^{3/2}] \\ & \times \left(-\frac{i}{2} A^2(t) k_{I,3} \sqrt{n(n+1)(n+2)} \langle q | \phi_{n+2} \rangle \right. \\ & + \left. \frac{i}{4} A^2(t) k_{I,3} \sqrt{(n-1)(n-2)(n-3)} \langle q | \phi_{n-4} \rangle \right. \\ & + \left. \frac{1}{2} [k_{R,1} \sqrt{n-1} + 3iA^2(t) k_{I,3} (n-1)^{3/2}] \langle q | \phi_{n-2} \rangle \right) \left. \right\}. \quad (\text{B4}) \end{aligned}$$